$$\tau_{1-2} = (M_2 - M_1)a^{3/2} \qquad (21)$$

Thus, using the above equations, all elements of the orbit can be obtained.

References

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A Supplementary Note on the Optimum Design of Box Beams for Combined Strength and Stiffness

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Reference 1 contains a discussion of some optimum-design factors for a wing box beam in which strength and stiffness requirements must be met.

Prior to this stage of design, and particularly in the case of swept wings, one is aware that the actual loads on the structure are functions of its torsional and bending stiffnesses. It follows that a judicious distribution of material can minimize the external loads on the wing. We have

$$M = \int_s^l \int_s^l l ds ds = \int_s^l \int_s^l \frac{qc}{144} (C_{L\alpha}\alpha_g + C_{L\alpha_e}\alpha_s) ds ds \qquad (1)$$

$$\alpha_s = \phi \cos\Lambda - \Gamma \sin\Lambda = \cos\Lambda \int_0^s \frac{T}{GJ}\, ds - \sin\Lambda \int_0^s \frac{M}{EI}\, ds \qquad (2)$$

and

$$T = \int_s^l l e_1 c ds \qquad (3)$$

It is seen from Eq. (1) that when α_s is a minimum, M is also a minimum. Since M and T are functions of α_s in the expression for α_s, it would be difficult to obtain directly a distribution of area so that α_s would be minimized at each station; and it is obvious that an iterative procedure is required in order that the final values of J, I, α_s, T, and M all be compatible in their relationships in the above equations. Furthermore, the distribution of material required for strength will not necessarily correspond to the distribution which yields minimum external loads.

It is seen from Eq. (2) that a high J and a low value of I would tend to minimize the incremental angle of attack under positive load conditions. We can write

$$\frac{GJ}{EI} = \frac{G}{E} \frac{4A^2/[(2c/t_s) + (2h/t_F)]}{(ct_e h^2)/2} \qquad (4)$$

and for $t_s = t_e$, this ratio becomes

$$\frac{GJ}{EI} = \frac{4G}{E} \frac{1}{1 + (t_s/t_F)(h/c)} \qquad (5)$$

This value is large when t_s/t_F and h/c are small.

Although it is difficult to distribute the material in the beam so that the bending moment is a minimum at *each* station, it is possible to distribute a fixed amount of material so that α_s is a minimum at the tip, or the material can be distributed so that α_s is a minimum at a specified station, at the same time keeping α_s at the tip at a fixed value.

$$\alpha_{sTIP} = \int_0^l \left[\frac{T(c+h)\cos\Lambda}{2GA^2 t} - \frac{2M \sin\Lambda}{Ecth^2}\right] ds = \int_0^l \frac{Bds}{t} \qquad (6)$$

and

$$W = \int_0^l \rho t p ds \qquad (7)$$

By the calculus of variations we find that the optimum distribution of t for minimum α_s, based on a constant value of B, is given by

$$t = \frac{W}{\rho\sqrt{pA}} \frac{\sqrt{\dfrac{T(c+h)\cos\Lambda}{2G} - \dfrac{2Mc\sin\Lambda}{E}}}{\displaystyle\int_0^l \sqrt{\frac{T(c+h)\cos\Lambda}{2GA^2} - \frac{2M\sin\Lambda}{Ech^2}}\sqrt{p}\, ds} \qquad (8)$$

The iterative process must, of course, be used to yield the distribution of t when B is adjusted to include the effects of loads due to deflection.

When α_s is to be a minimum at a given station and fixed at the tip, we have

$$\alpha_{s_1} + \alpha_{s_2} = \alpha_{sTIP} \quad \text{(fixed)} \qquad (9)$$

and

$$W_1 + W_2 = W \quad \text{(fixed)} \qquad (10)$$

Under the conditions given in Eqs. (9) and (10), α_{s_1} is to be minimum for the available W_1, while α_{s_2} is the allowable remaining deflection to meet the given tip deflection, α_{sTIP}, for a minimum W_2. Application of the Calculus of Variations yields two simultaneous equations in the Lagrange multipliers, each corresponding to one of the beam sections:

$$\left[\sqrt{p}\int_0^{s_1}\sqrt{B}\sqrt{p}\, ds\right]\sqrt{\lambda_1} + \left[\sqrt{p}\int_{s_1}^l \sqrt{B}\sqrt{p}\, ds\right]\sqrt{\lambda_2} = \alpha_{sTIP} \qquad (11)$$

$$\left[\sqrt{p}\int_0^{s_1}\sqrt{B}\sqrt{p}\, ds\right]\frac{1}{\sqrt{\lambda_1}}\left[\sqrt{p} + \int_{s_1}^l \sqrt{B}\sqrt{p}\, ds\right]\frac{1}{\sqrt{\lambda_2}} = W \qquad (12)$$

The solution of Eqs. (11) and (12) gives two values of λ_1, one of which is a minimum and the other a maximum, both compatible with the fixed tip deflection and total weight; at the same time the most efficient use of material in each segment is realized. Again, iteration must be used in the process in order to reflect the effects on the value of loads due to deflections, and the resulting quadratic equation must be solved for each iteration. The numerical values of the multipliers are used in the solution for the optimum distribution of material in each section. Ref. 2 discusses the use of the multiplier in a single-section beam.

Much attention has recently been given to the redundant analysis of internal-load distributions; however, it should also be noted that the stiffness distribution in a structure will affect the external loads.

References

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